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THE RENORMALIZATION GROUP EQUATION IN N=2 SUPERSYMMETRIC GAUGE THEORIES*

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Abstract

We clarify the mass dependence of the effective prepotential in N=2 supersymmetric SU(N_c) gauge theories with an arbitrary number $N_f < 2N_c$ of flavors. The resulting differential equation for the prepotential extends the equations obtained previously for SU(2) and for zero masses. It can be viewed as an exact renormalization group equation for the prepotential, with the beta function given by a modular form. We derive an explicit formula for this modular form when $N_f = 0$, and verify the equation to 2-instanton order in the weak-coupling regime for arbitrary N_f and N_c.

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I. INTRODUCTION

New avenues for the investigation of N=2 supersymmetric gauge theories have recently opened up with the Seiberg-Witten proposal [1], which gives the effective action in terms of a 1-form $\mathrm{d}\lambda$ on Riemann surfaces fibering over the moduli space of vacua. Starting with the SU(2) theory [1], a form $\mathrm{d}\lambda$ is now available for many other gauge groups [2], with matter in the fundamental [3][4] or in the adjoint representation [5]. This has led to a wealth of information about the prepotential, including its expansion up to 2-instanton order for asymptotically free theories with classical gauge groups [6].

These developments suggest a rich structure for the prepotential $\mathcal{F}$, which may help understand its strong coupling behavior, and clarify its relation with the point particle limit of string theories, when gravity is turned off [7]. Of particular interest in this context are the non-perturbative differential equations derived by Matone in [8] for SU(2), and later extended by Eguchi and Yang in [9] to SU(N_c) theories with only massless matter. It was however unclear how these equations would be affected if the hypermultiplets acquire non-vanishing masses.

In the present paper, we address this issue by providing a systematic and general framework for incorporating arbitrary masses m_j. In effect, the masses m_j are treated on an equal footing as the vev's a_k of the scalar field in the chiral multiplet, since they are both given by periods of $\mathrm{d}\lambda$ around non-trivial cycles. For the masses, the cycles are small loops around the poles of $\mathrm{d}\lambda$, while for a_k, they are non-trivial A-homology cycles. This suggests that the derivatives of $\mathcal{F}$ with respect to the masses should be given by the periods of $\mathrm{d}\lambda$ around "dual cycles", just as the derivatives of $\mathcal{F}$ with respect to a_k are given by the periods of $\mathrm{d}\lambda$ around B-cycles. We provide an explicit closed formula for such a prepotential, motivated by the τ-function of the Whitham hierarchy obtained in [10]. (In this connection, we should point out that intriguing similarities between supersymmetric gauge theories and Whitham hierarchies had been noted by many authors [11], and had been the basis of the considerations in [9], as well as in [4], the starting point of our arguments). Written in terms of the derivatives of $\mathcal{F}$, this closed formula becomes the non-perturbative equation for $\mathcal{F}$ that we seek. It can be verified explicitly to 2-instanton order, using the results of [6].

Specifically, the differential equation for $\mathcal{F}$ is of the form

$$\mathcal{D}\mathcal{F} = -\frac{1}{2\pi i}\big[\mathrm{Res}_{P_-}(z\mathrm{d}\lambda)\mathrm{Res}_{P_-}(z^{-1}\mathrm{d}\lambda) + \mathrm{Res}_{P_+}(z\mathrm{d}\lambda)\mathrm{Res}_{P_+}(z^{-1}\mathrm{d}\lambda)\big] \tag{1.1}$$

with $\mathcal{D}$ the operator

$$\mathcal{D} = \sum_{k=1}^{N_c} a_k \frac{\partial}{\partial a_k} + \sum_{j=1}^{N_f} m_j \frac{\partial}{\partial m_j} - 2 \tag{1.2}$$

The right hand side in (1.1) has been interpreted in [8][9] in terms of the trace of the classical vacuum expectation value $\sum_{k=1}^{N_c} \bar{a}_k^2$, although there are ambiguities with this interpretation when $N_f \geq N_c$. Mathematically, it can be expressed in terms of ϑ-functions for arbitrary N_c when $N_f = 0$ (c.f. Section III (c) below). There is little doubt that this should be the case in general. Now we have by dimensional analysis

$$(\mathcal{D} + \Lambda \frac{\partial}{\partial \Lambda})\mathcal{F} = 0 \tag{1.3}$$

if Λ is the renormalization scale of the theory. Thus the proper interpretation for the equation (1.1) is as a renormalization group equation, with the beta function given by a modular form!

Finally, we observe that the effective Lagrangian in the low momentum expansion determines the effective prepotential only up to a_k-independent terms. However, masses can arise as vacuum expectation values of non-dynamical fields, and we would expect the natural dependence on masses imposed here to be useful in future developments, for example in eventual generalizations to string theories.

II. A CLOSED FORM FOR THE PREPOTENTIAL

(a) The geometric set-up for N=2 supersymmetric gauge theories

We recall the basic set-up for the effective prepotential $\mathcal{F}$ of N=2 supersymmetric SU(N_c) gauge theories.

The moduli space of vacua is an $N_c - 1$ dimensional variety, which can be parametrized classically by the eigenvalues $\bar{a}_k$, $\sum_{k=1}^{N_c} \bar{a}_k = 0$ of the scalar field ϕ in the adjoint representation occurring in the N=2 chiral multiplet. (The flatness of the potential is equivalent to $[\phi, \phi^\dagger] = 0$). Quantum mechanically, the order parameters $\bar{a}_k$ get renormalized to parameters a_k. The prepotential $\mathcal{F}$ determines completely the Wilson effective Lagrangian of the quantum theory to leading order in the low momentum expansion. Following Seiberg-Witten [1], we require that the renormalized order parameters a_k, their duals $a_{D,k}$, and the prepotential $\mathcal{F}$ be given by

$$\begin{aligned} a_k &= \frac{1}{2\pi i} \oint_{A_k} d\lambda, \qquad a_{D,k} = \frac{1}{2\pi i} \oint_{B_k} d\lambda \\ \frac{\partial \mathcal{F}}{\partial a_k} &= a_{D,k} \end{aligned} \tag{2.1}$$

where $d\lambda$ is a suitably chosen meromorphic 1-form on a fibration of Riemann surfaces Γ above the moduli space of vacua, and A_j, B_j is a canonical basis of homology cycles on Γ.

In the formalism of [4], the form $\mathrm{d}\lambda$ is characterized by two meromorphic Abelian differentials $\mathrm{d}Q$ and $\mathrm{d}E$ on Γ, with $\mathrm{d}\lambda = Q\mathrm{d}E$. For $\mathrm{SU}(N_c)$ gauge theories with N_f hypermultiplets in the fundamental representation, $N_f < 2N_c$, the defining properties of $\mathrm{d}E$ and $\mathrm{d}Q$ are

- $\mathrm{d}E$ has only simple poles, at points P_+, P_-, P_i, where its residues are respectively $-N_c$, $N_c - N_f$, and 1 $(1 \leq i \leq N_f)$. Its periods around homology cycles are integer multiples of $2\pi i$;
- Q is a well-defined meromorphic *function*, which has simple poles at P_+ and P_-, and takes the values $Q(P_i) = -m_i$ at P_i, where m_i are the bare masses of the N_f hypermultiplets;
- The form $\mathrm{d}\lambda$ is normalized so that

$$\mathrm{Res}_{P_+}(z\mathrm{d}\lambda) = -N_c 2^{-1/N_c},\ \mathrm{Res}_{P_-}(z\mathrm{d}\lambda) = (N_c - N_f)(\frac{\Lambda^{2N_c-N_f}}{2})^{1/(N_c-N_f)}$$
$$\mathrm{Res}_{P_+}(\mathrm{d}\lambda) = 0 \tag{2.2}$$

  where Λ is the dynamically generated scale of the theory, and $z = E^{-1/N_c}$ or $z = E^{1/(N_c-N_f)}$ is the holomorphic coordinate system provided by the Abelian integral E, depending on whether we are near P_+ or near P_-.

It was shown in [4] that these conditions imply that Γ is hyperelliptic, and admits an equation of the form

$$y^2 = \Big(\prod_{k=1}^{N_c}(Q - \tilde{a}_k)\Big)^2 - \Lambda^{2N_c-N_f}\prod_{j=1}^{N_f}(Q + m_j) \equiv A(Q)^2 - B(Q) \tag{2.3}$$

Here $\tilde{a}_k$ are parameters which coincide with $\bar{a}_k$ when $N_c < N_f$, but may otherwise receive corrections. It is convenient to set

$$\bar{\Lambda} = \Lambda^{\frac{1}{2}(2N_c-N_f)}$$

The function Q in $\mathrm{d}\lambda = Q\mathrm{d}E$ is now the coordinate Q in the complex plane, lifted to the two sheets $y = \pm\sqrt{A^2 - B}$ of (2.3), while the Abelian integral E is given by $E = \log(y + A(Q))$. The points $P_\pm$ correspond to $Q = \infty$, with the choice of signs $y = \pm\sqrt{A^2 - B}$.

(b) The prepotential in closed form

We shall now exhibit a solution $\mathcal{F}$ for the equations (2.1) in closed form. Formally, it is given by

$$2\mathcal{F} = \frac{1}{2\pi i}\Big[\sum_{k=1}^{N_c} a_k \oint_{B_k}\mathrm{d}\lambda - \sum_{j=1}^{N_f} m_j \int_{P_-}^{P_j}\mathrm{d}\lambda$$
$$+ \mathrm{Res}_{P_+}(z\mathrm{d}\lambda)\mathrm{Res}_{P_+}(z^{-1}\mathrm{d}\lambda) + \mathrm{Res}_{P_-}(z\mathrm{d}\lambda)\mathrm{Res}_{P_-}(z^{-1}\mathrm{d}\lambda)\Big] \tag{2.4}$$

However, the above expression involves divergent integrals which must be regularized. For this, we need to make a number of choices. First, we fix a canonical homology basis A_i, B_i, along which the Riemann surface can be cut out to obtain a domain with boundary $\prod_{i=1}^{N_c-1} A_i^{-1}B_i^{-1}A_iB_i$. Next, we fix simple paths C_-, C_j from P_+ to P_-, P_j respectively ($1 \leq j \leq N_f$), which have only P_+ as common point. As usual the cuts are viewed as having two edges. With these choices, we can define a single-valued branch of the Abelian integral E in $\Gamma_{cut} = \Gamma \setminus (C_- \cup C_1 \cup \cdots \cup C_{N_f})$ as follows. Near P_+, the function Q^{-1} provides a biholomorphism of a neighborhood of P_+ to a small disk in the complex plane. Choose the branch of $\log Q^{-1}$ with a cut along $Q^{-1}(C_-)$, and define an integral E of $\mathrm{d}E$ in a neighborhood of P_+ in Γ_{cut} by requiring that

$$E = N_c \log Q + \log 2 + O(Q^{-1}) \tag{2.5}$$

The Abelian integral E can then uniquely defined on Γ_{cut} by integrating along paths. It determines in turn a coordinate system z near each of the poles P_+, P_-, and P_j, $1 \leq j \leq N_f$, e.g.,

$$z = e^{-\frac{1}{N_c}E} \quad \text{near} \quad P_+ \tag{2.6}$$

It is easily seen that z is holomorphic around P_+, and that $z = 2^{\frac{1}{N_c}}Q^{-1} + O(Q^{-2})$. The next few terms of the expansion of z in terms of Q^{-1} are actually quite important, but we shall evaluate them later. Similarly, we set $z = e^{\frac{1}{N_c-N_f}E}$ near P_-, and $z = e^{-E}$ near P_j, $1 \leq j \leq N_f$.

The same choices above allow us to define at the same time a single-valued branch of the Abelian integral λ in Γ_{cut}. Specifically, λ is defined near P_+ by the normalization

$$\lambda(z) = -\mathrm{Res}_{P_+}(z\mathrm{d}\lambda)\frac{1}{z} + O(z) \tag{2.7}$$

with z the above holomorphic coordinate (2.6). As before, λ is then extended to the whole of Γ_{cut} by analytic continuation. Evidently, near P_-, λ can be expressed as

$$\lambda(z) = -\mathrm{Res}_{P_-}(z\mathrm{d}\lambda)\frac{1}{z} + \lambda(P_-) + O(z) \tag{2.8}$$

in the corresponding coordinate z near P_-, for a suitable constant $\lambda(P_-)$. Similarly, near P_j, λ can be expressed as

$$\lambda(z) = -m_j \log z + \lambda(P_j) + O(z) \tag{2.9}$$

for suitable constants P_j. The expression (2.4) for the prepotential $\mathcal{F}$ can now be given a precise meaning by regularizing as follows the divergent integrals appearing there

$$\int_{P_-}^{P_j} \mathrm{d}\lambda = \lambda(P_j) - \lambda(P_-) \tag{2.10}$$

This method of regularization has the advantage of commuting with differentiation under the integral sign with respect to connections which keep the values of z constant.

(c) The derivatives of the prepotential

The main properties of $\mathcal{F}$ are the following

$$\frac{\partial\mathcal{F}}{\partial a_k} = \frac{1}{2\pi i}\oint_{B_k} \mathrm{d}\lambda \tag{2.11}$$

$$\frac{\partial\mathcal{F}}{\partial m_j} = \frac{1}{2\pi i}\Big[-\int_{P_-}^{P_j} \mathrm{d}\lambda + \frac{1}{2}\sum_{i=1}^{N_f} m_i\big(\int_{P_-}^{P_i} \mathrm{d}\Omega_j^{(3)} - \int_{P_-}^{P_j} \mathrm{d}\Omega_i^{(3)}\big)\Big] \tag{2.12}$$

where $\mathrm{d}\Omega_i^{(3)}$ are Abelian differentials of the third kind with simple poles and residues +1 and -1 at P_- and P_i respectively, normalized to have vanishing A_j-periods. We observe that the Wilson effective action of the gauge theory is insensitive to modifications of $\mathcal{F}$ by a_k-independent terms. The equation (2.12) can be viewed as an additional criterion for selecting $\mathcal{F}$, motivated by the fact that the mass parameter $-m_j$ of $\mathrm{d}\lambda$ can be viewed as a contour integral of $\mathrm{d}\lambda$ around a cycle surrounding the pole P_j. In analogy with (2.4), the derivatives with respect to m_j of a natural choice for $\mathcal{F}$ should then reproduce the integral of $\mathrm{d}\lambda$ around a dual cycle. This is the origin of the first term on the right hand side of (2.12), if we view the path from P_- to P_j as such a dual "cycle". The second term on the right hand side of (2.12) is a harmless correction due to regularization. The expression between parentheses is actually always a multiple of πi, although we do not need this fact.

We now establish (2.11) and (2.12). We need to consider the derivatives of $\mathrm{d}\lambda$ with respect to both a_k and m_j. We use the connection $\nabla^E = \nabla$ of [4], which differentiates along subvarieties where the value of the Abelian integral E (equivalently the coordinate z) is kept constant. Then simply by inspecting the derivatives of the singular parts of $\mathrm{d}\lambda$ in a Laurent expansion in the z-coordinate near each pole, we find that

$$\nabla_{a_k}\mathrm{d}\lambda = 2\pi i\,\mathrm{d}\omega_k,\ \nabla_{m_j}\mathrm{d}\lambda = \mathrm{d}\Omega_j^{(3)}, \tag{2.13}$$

where $\mathrm{d}\omega_k$ is the basis of Abelian differentials of the first kind dual to the A_k-cycles. Next, we recall from (2.2) that the residues $\mathrm{Res}_{P_+}(z\mathrm{d}\lambda)$ and $\mathrm{Res}_{P_-}(z\mathrm{d}\lambda)$ are constant. Consequently,

$$2\frac{\partial\mathcal{F}}{\partial a_k} = \frac{1}{2\pi i}\oint_{B_k} \mathrm{d}\lambda + \sum_{i=1}^{N_c} a_i \oint_{B_i} \mathrm{d}\omega_k - \sum_{j=1}^{N_f} m_j \int_{P_-}^{P_j} \mathrm{d}\omega_k$$
$$+ \mathrm{Res}_{P_+}(z\mathrm{d}\lambda)\mathrm{Res}_{P_+}(z^{-1}\mathrm{d}\omega_k) + \mathrm{Res}_{P_-}(z\mathrm{d}\lambda)\mathrm{Res}_{P_-}(z^{-1}\mathrm{d}\omega_k) \tag{2.14}$$

However, we also have the following Riemann bilinear relations, valid even in presence of regularizations

$$\begin{aligned}
\oint_{B_i} \mathrm{d}\omega_k &= \oint_{B_k} \mathrm{d}\omega_i, \\
\frac{1}{2\pi i}\oint_{B_k} \mathrm{d}\Omega_j^{(3)} &= -\int_{P_-}^{P_j} \mathrm{d}\omega_k \\
\frac{1}{2\pi i}\oint_{B_k} \mathrm{d}\Omega_\pm^{(2)} &= \mathrm{Res}_{P_\pm}(z^{-1}\mathrm{d}\omega_k) \\
\int_{P_-}^{P_j} \mathrm{d}\Omega_\pm^{(2)} &= -\mathrm{Res}_{P_\pm}(z^{-1}\mathrm{d}\Omega_j^{(3)})
\end{aligned} \tag{2.15}$$

Here $\mathrm{d}\Omega_\pm^{(2)}$ are Abelian differentials of the second kind, with a double pole at $P_\pm$, vanishing A-cycles, and normalization

$$\mathrm{d}\Omega_\pm^{(2)} = z^{-2}\mathrm{d}z + O(z) \tag{2.16}$$

The relations (2.15) follow from the usual Riemann bilinear arguments, by considering respectively the (vanishing) integrals on the cut surface Γ_{cut} of the 2-forms $\mathrm{d}(\omega_i \mathrm{d}\omega_k)$, $\mathrm{d}(\Omega_j^{(3)}\mathrm{d}\omega_k)$, $\mathrm{d}(\Omega_\pm^{(2)}\mathrm{d}\omega_k)$, $\mathrm{d}(\Omega_j^{(3)}\mathrm{d}\Omega_\pm)$. Applying (2.15) to (2.14), we obtain

$$\begin{aligned}
2\frac{\partial \mathcal{F}}{\partial a_k} =& \frac{1}{2\pi i}\oint_{B_k} \mathrm{d}\lambda + \sum_{i=1}^{N_c} a_i \oint_{B_k} \mathrm{d}\omega_i + \frac{1}{2\pi i}\sum_{j=1}^{N_f} m_j \oint_{B_k} \mathrm{d}\Omega_j^{(3)} \\
&+ \frac{1}{2\pi i}\mathrm{Res}_{P_+}(z\mathrm{d}\lambda)\oint_{B_k} \mathrm{d}\Omega_+^{(2)} + \frac{1}{2\pi i}\mathrm{Res}_{P_-}(z\mathrm{d}\lambda)\oint_{B_k} \mathrm{d}\Omega_-^{(2)}
\end{aligned} \tag{2.17}$$

However, the expression

$$\mathrm{d}\lambda = 2\pi i \sum_{i=1}^{N_c} a_i \mathrm{d}\omega_i + \mathrm{Res}_{P_+}(z\mathrm{d}\lambda)\mathrm{d}\Omega_+^{(2)} + \mathrm{Res}_{P_-}(z\mathrm{d}\lambda)\mathrm{d}\Omega_-^{(2)} + \sum_{j=1}^{N_f} m_j \mathrm{d}\Omega_j^{(3)} \tag{2.18}$$

is just the expansion of $\mathrm{d}\lambda$ in terms of Abelian differentials of first, second, and third kind! The equation (2.11) follows. The equation (2.12) can be established in the same way. First we write

$$\begin{aligned}
2\frac{\partial \mathcal{F}}{\partial m_l} =& \frac{1}{2\pi i}\Big[\sum_{i=1}^{N_c} a_i \oint_{B_i} \mathrm{d}\Omega_l^{(3)} - \int_{P_-}^{P_l} \mathrm{d}\lambda - \sum_{j=1}^{N_f} m_j \int_{P_-}^{P_j} \mathrm{d}\Omega_l^{(3)} \\
&+ \mathrm{Res}_{P_+}(z\mathrm{d}\lambda)\mathrm{Res}_{P_+}(z^{-1}\mathrm{d}\Omega_l^{(3)}) + \mathrm{Res}_{P_-}(z\mathrm{d}\lambda)\mathrm{Res}_{P_-}(z^{-1}\mathrm{d}\Omega_l^{(3)})\Big]
\end{aligned} \tag{2.19}$$

Substituting in the bilinear relations gives

$$2\frac{\partial \mathcal{F}}{\partial m_l} = \frac{1}{2\pi i}\Big[-2\pi i\sum_{i=1}^{N_c} a_i\int_{P_-}^{P_l} \mathrm{d}\omega_l - \int_{P_-}^{P_l} \mathrm{d}\lambda - \sum_{j=1}^{N_f} m_j\int_{P_-}^{P_l} \mathrm{d}\Omega_j^{(3)}$$
$$- \mathrm{Res}_{P_+}(z\mathrm{d}\lambda)\int_{P_-}^{P_l} \mathrm{d}\Omega_+^{(2)} - \mathrm{Res}_{P_-}(z\mathrm{d}\lambda)\int_{P_-}^{P_l} \mathrm{d}\Omega_-^{(2)}\Big]$$
$$- \frac{1}{2\pi i}\sum_{j=1}^{N_f} m_j\Big(\int_{P_-}^{P_j} \mathrm{d}\Omega_l^{(3)} - \int_{P_-}^{P_l} \mathrm{d}\Omega_j^{(3)}\Big) \tag{2.20}$$

Again, the Abelian differentials recombine to produce $\mathrm{d}\lambda$, and the relation (2.12) follows.

III. THE RENORMALIZATION GROUP EQUATION

(a) The renormalization group equation in terms of residues

Combining the equations (2.4), (2.11), and (2.12) gives a first version of the renormalization group equation for $\mathcal{F}$, valid in presence of arbitrary masses m_j

$$\sum_{k=1}^{N_c} a_k\frac{\partial \mathcal{F}}{\partial a_k} + \sum_{j=1}^{N_f} m_j\frac{\partial \mathcal{F}}{\partial m_j} - 2\mathcal{F} = -\frac{1}{2\pi i}\Big[\mathrm{Res}_{P_+}(z\mathrm{d}\lambda)\mathrm{Res}_{P_+}(z^{-1}\mathrm{d}\lambda)$$
$$+ \mathrm{Res}_{P_-}(z\mathrm{d}\lambda)\mathrm{Res}_{P_-}(z^{-1}\mathrm{d}\lambda)\Big] \tag{3.1}$$

(b) The renormalization group equation in terms of invariant polynomials

We can evaluate the right hand side of (3.1) explicitly, in terms of the masses m_j, and the moduli parameters $\tilde{a}_k$ and Λ of the spectral curve (2.3). For this, we need the first three leading coefficients in the expansion of Q in terms of z at P_+ and P_-. Now recall that at P_+, $Q \to \infty$, $y = \sqrt{A^2 - B}$, and

$$z = (y + A)^{-1/N_c} \tag{3.2}$$

For $N_f < 2N_c$, we may expand $\sqrt{A^2 - B}$ in powers of B/A^2 and write, up to $O(Q^{N_c-3})$

$$y + A = 2\big[A - \frac{1}{4}\frac{B}{A} - \frac{1}{16}\frac{B^2}{A^3}\big] \tag{3.3}$$

We consider first the terms in (3.3) of order up to $O(Q^{N_c-1})$. Then for $N_f \le 2N_c - 2$, only the top two terms in A contribute, while for $N_f = 2N_c - 1$, we must also incorporate the term $\bar{\Lambda}^2 x^{N_f - N_c} = \bar{\Lambda}^2 x^{N_c-1}$ from B/A. Thus

$$A + y = 2Q^{N_c}\big[1 - (\tilde{s}_1 + \delta_{N_f,2N_c-1}\frac{\bar{\Lambda}^2}{4})Q^{-1}\big] + O(Q^{N_c-2})$$

where we have introduced the notation

$$\tilde{s}_i = (-1)^i \sum_{k_1 < \cdots < k_i} \tilde{a}_{k_1} \cdots \tilde{a}_{k_i}, \qquad t_i = \sum_{k_1 < \cdots < k_i} \tilde{t}_{k_1} \cdots \tilde{t}_{k_i}$$

This leads to the first two coefficients of z in terms of Q, or equivalently, the first two coefficients of Q in terms of z

$$Q = 2^{-1/N_c} z^{-1} \big(1 + \frac{1}{N_c} (\tilde{s}_1 + \delta_{N_f, 2N_c - 1} \frac{\bar{\Lambda}^2}{4}) z \big)$$

Comparing with (2.2), we see that this confirms the value of $\mathrm{Res}_{P_+}(z \mathrm{d}\lambda)$ required there, while the condition that $\mathrm{Res}_{P_+}(\mathrm{d}\lambda) = 0$ is equivalent to

$$\tilde{s}_1 + \delta_{N_f, 2N_c - 1} \frac{\bar{\Lambda}^2}{4} = 0 \tag{3.4}$$

Similarly, in the expansion of $A + y$ to order $O(Q^{N_c - 2})$, we must consider separately the cases $N_f < 2N_c - 2$, $N_f = 2N_c - 2$, and $N_f = 2N_c - 1$, depending on whether the terms B/A and B^2/A^3 contribute to this order. Taking into account (3.4), we find

$$Q = 2^{-1/N_c} z^{-1} \big(1 - \frac{2^{2/N_c}}{N_c} S_2^+ z^2 \big) + O(z^2) \tag{3.5}$$

with S_2^+ defined to be

$$S_2^+ = \tilde{s}_2 - \delta_{N_f, 2N_2 - 2} \frac{\bar{\Lambda}^2}{4} - \delta_{N_f, 2N_c - 1} \frac{\bar{\Lambda}^2}{4} t_1 \tag{3.6}$$

Near P_-, we have instead

$$A + y = A - A(1 - \frac{B}{A^2})^{1/2} = \frac{1}{2} \frac{B}{A} + \frac{1}{8} \frac{B^2}{A^3} + \frac{1}{16} \frac{B^3}{A^5}$$

Again, considering separately the cases $N_f < 2N_c - 2$, $N_f = 2N_c - 2$, $N_f = 2N_c - 1$, we can derive the leading three terms of the expansion of $z = E^{-1/(N_f - N_c)} = (A + y)^{-1/(N_f - N_c)}$ in terms of Q. Written in terms of an expansion of Q in terms of z, the result is

$$\begin{aligned} Q =& (\frac{\bar{\Lambda}^2}{2})^{-1/(N_f - N_c)} z^{-1} \Big[1 - \frac{t_1}{N_f - N_c} (\frac{\bar{\Lambda}^2}{2})^{1/(N_f - N_c)} z \\ &+ \frac{1}{(N_f - N_c)^2} (\frac{\bar{\Lambda}^2}{2})^{2/(N_f - N_c)} \big(S_2^- (N_f - N_c) + \frac{1}{2} (-1 + N_f - N_c) t_1^2 \big) z^2 \Big] \end{aligned} \tag{3.7}$$

with S_2^- given by

$$S_2^- = \tilde{s}_2 - t_2 - \delta_{N_f, 2N_2 - 2} \frac{\bar{\Lambda}^2}{4} - \delta_{N_f, 2N_c - 1} \frac{\bar{\Lambda}^2}{4} t_1 \tag{3.8}$$

Since $\mathrm{d}\lambda = -N_c Q \frac{\mathrm{d}z}{z}$ near P_+ and $\mathrm{d}\lambda = -(N_f - N_c) Q \frac{\mathrm{d}z}{z}$ near P_-, we obtain

$$\begin{aligned}
\mathrm{Res}_{P_+}(z^{-1}\mathrm{d}\lambda) =& 2^{1/N_c} S_2^+, \\
\mathrm{Res}_{P_-}(z^{-1}\mathrm{d}\lambda) =& -(\frac{\bar\Lambda^2}{2})^{1/(N_f-N_c)} \big(S_2^- + \frac{1}{2}(1 - \frac{1}{N_f - N_c}) t_1^2\big)
\end{aligned} \tag{3.9}$$

Subsituting in the values of $\mathrm{Res}_{P_+}(z\mathrm{d}\lambda)$ and $\mathrm{Res}_{P_-}(z\mathrm{d}\lambda)$ given in (2.2), and rewriting the result in terms of $\tilde s_2$ and the operator $\mathcal{D}$ of (1.2), we can rewrite the renormalization group equation (3.1) as

$$\begin{aligned}
2\pi i \mathcal{D}\mathcal{F} = -(N_f - 2N_c)\big\{\tilde s_2 - \delta_{N_f, 2N_c - 2}\frac{\bar\Lambda^2}{4} - \delta_{N_f, 2N_c-1}\frac{\bar\Lambda^2}{4} t_1\big\} \\
+ (N_f - N_c) t_2 - \frac{1}{2}(N_f - N_c - 1) t_1^2
\end{aligned} \tag{3.10}$$

Before proceeding further, we would like to note a few features of the renormalization group equation and of our choice of prepotential.

(1) The RG equations (3.1) and (3.10) are actually invariant under a change of cuts. Indeed, a change of cuts would shift the values of the regularized integrals (1.4) by a linear expression, and hence $\mathcal{F}$ by a quadratic expression in the masses m_j, independent of the a_k. In view of Euler's relation, such terms cancel in the left hand side of (3.1) and (3.10). Thus the right hand side of the RG only transforms under a change of homology basis, and is a modular form;

(2) From the point of view of gauge theories alone, we can in practice ignore on the right hand side of (3.1) and (3.10) terms which do not depend on the a_k. Such terms can always be cancelled by a suitable a_k-independent correction to $\mathcal{F}$. These corrections do not affect the Wilson effective action since it depends only on the derivatives of $\mathcal{F}$ with respect to a_k;

(3) Some caution may be necessary in interpreting $\tilde s_2$, in terms of the classical order parameters $\bar a_k$. In particular, when $N_f \geq N_c$, there are several natural ways of parametrizing the curve (2.3), which the $\tilde a_k$ get shifted in different ways to $\tilde a_k \neq a_k$ [3][4]. As noted in [6], the prepotential $\mathcal{F}$ is independent of such redefinitions of the $\bar a_k$. However, this would of course not be the case for $\bar s_2 \equiv \sum_{k<j}^{N_c} \bar a_k \bar a_j$, which argues for a distinct interpretation for $\tilde s_2 = \sum_{j<k} \tilde a_k \tilde a_j$.

(c) The renormalization group equation in terms of ϑ-functions

As noted above, the right hand side of the RG equation (3.1) is in general a modular form. For $N_f = 0$ (and arbitrary N_c), we can exploit the symmetry between the branch points $x_k^\pm$ given by $y^2 = (A - \bar\Lambda)(A + \bar\Lambda) = \prod_{k=1}^{N_c}(Q - x_k^+)(Q - x_k^-)$ and known formulae for

their cross-ratios to write it explicitly in terms of ϑ-functions. More precisely, we observe that

$$\sum_{k=1}^{N_c} \tilde{a}_k^2 = \sum_{k=1}^{N_c} (x_k^+)^2 = \sum_{k=1}^{N_c} (x_k^+)^2 \tag{3.11}$$

Let the canonical homology basis be given by A_k cycles surrounding the cut from x_k^- to x_k^+, $1 \leq k \leq N_c - 1$ on one sheet, and by B_k cycles going from $x_{N_c}^-$ to x_k^- on one sheet, and coming back from x_k^- to $x_{N_c}^-$ on the opposite sheet. Then for the dual basis of Abelian differentials $\mathrm{d}\omega = (\mathrm{d}\omega_k)_{k=1,\cdots,N_c-1}$, we introduce the basis vectors $e^{(k)}$ and $\tau^{(k)}$ of the Jacobian lattice by

$$\oint_{A_k} \mathrm{d}\omega = e^{(k)}, \quad \oint_{B_k} \mathrm{d}\omega = \tau^{(k)}$$

We have then the following relations between points in the Jacobian lattice

$$\int_{x_k^-}^{x_k^+} \mathrm{d}\omega = \frac{1}{2} e^{(k)}, \quad \int_{x_k^+}^{x_{k+1}^-} \mathrm{d}\omega = \frac{1}{2}(\tau^{(k+1)} + \tau^{(k)}) \tag{3.12}$$

Let $\phi(Q)$ denote the Abel map

$$\phi(P) = (\int_{Q_0}^{Q} \mathrm{d}\omega_1, \cdots, \int_{Q_0}^{Q} \mathrm{d}\omega_{N_c-1})$$

If we choose Q_0 so that $\phi(x_1^-) = \frac{1}{2}\tau^{(1)}$, it follows from (3.12) that

$$\begin{aligned}
\phi(x_k^-) =& \frac{1}{2}(e^{(1)} + \cdots + e^{(k-1)}) + \frac{1}{2}\tau^{(k)}, \ 1 \leq k \leq N_c - 1 \\
\phi(x_k^+) =& \frac{1}{2}(e^{(1)} + \cdots + e^{(k)}) + \frac{1}{2}\tau^{(k)}, \ 1 \leq k \leq N_c - 1 \\
\phi(x_{N_c}^-) =& \frac{1}{2}(e^{(1)} + \cdots + e^{(N_c-1)} \\
\phi(x_{N_c}^+) =& 0
\end{aligned} \tag{3.13}$$

If we introduce the functions $F_l^k(Q)$ by

$$F_l^k(Q) = \frac{\vartheta(\phi(x_l^- + x_k^+ + Q)|\tau)^2}{\vartheta(\phi(x_{N_c}^- + x_k^+ + Q)|\tau)^2} \tag{3.14}$$

an inspection of the zeroes shows that we have the following relation between F_l^k and cross-ratios

$$\frac{F_l^k(Q')}{F_l^k(Q)} = \frac{Q' - x_l^-}{Q - x_l^-} \frac{Q - x_{N_c}^-}{Q' - x_{N_c}^-} \tag{3.15}$$

For the Riemann surface (2.2), we also have for all Q

$$\prod_{l=1}^{N_c}(Q - x_l^+) = A(Q) - \bar{\Lambda} = \prod_{l=1}^{N_c}(Q - x_l^-) - 2\bar{\Lambda} \tag{3.16}$$

Setting $Q = x_k^+$ gives the relation

$$\prod_{l=1}^{N_c}(x_k^+ - x_l^-) = 2\bar{\Lambda} \tag{3.17}$$

Combining with products of expressions of the form (3.15) evaluated at branch points, we can actually identify the branch points

$$\begin{aligned} x_k^+ - x_{N_c}^- =& \Lambda G_k \\ x_k^+ - x_l^+ =& \Lambda(G_k - G_l) \\ x_k^+ =& -\frac{\Lambda}{N_c}\sum_{l=1}^{N_c} G_l + \Lambda G_k \end{aligned} \tag{3.18}$$

where G_k is defined to be

$$G_k = 2^{\frac{1}{N_c}} \prod_{l=1}^{N_c} \left\{ \left[\frac{F_l^k(x_m^-)}{F_l^k(x_k^+)}\right]^{\frac{1}{N_c}} \prod_{k'=1}^{N_c} \left[\frac{F_l^{k'}(x_{k'}^+)}{F_l^{k'}(x_m^-)}\right]^{\frac{1}{N_c^2}} \right\} \tag{3.19}$$

Since $F_l^k(x_k^-)$ is independent of k, this expression may be simplified,

$$G_k = 2^{\frac{1}{N_c}} \prod_{l=1}^{N_c} \prod_{k'=1}^{N_c} \left[\frac{F_l^k(x_m^-)}{F_l^{k'}(x_m^-)}\right]^{\frac{1}{N_c^2}} \tag{3.20}$$

The evaluation of the functions F_l^k on the branch points is particularly simple, and we have

$$F_l^k(x_m^-) = \frac{\vartheta(\phi(x_l^-) + \phi(x_m^-) + \phi(x_k^+)|\tau)^2}{\vartheta(\phi(x_{N_c}^-) + \phi(x_m^-) + \phi(x_k^+)|\tau)^2} \tag{3.21}$$

where the values of $\phi(x^\pm)$ can be read off from (3.13). This leads to the following expression for the right hand side of (3.10) :

$$\sum_{k=1}^{N_c} \tilde{a}_k^2 = \Lambda^2 \sum_{k=1}^{N_c} Q_k^2 - \frac{\Lambda^2}{N_c}\Big(\sum_{k=1}^{N_c} Q_k\Big)^2 \tag{3.22}$$

which is a modular form.

IV. THE WEAK-COUPLING LIMIT

It is instructive to verify the renormalization group equation (3.10) in the weak-coupling limit analyzed in [6] to 2-instanton order.

We recall the expression obtained in [6] for the prepotential $\mathcal{F}$ to two-instanton order in the regime of $\Lambda \to 0$. Let the functions $S_(x)$ and $S_k(x)$ be defined by

$$S(x) = \frac{\prod_{j=1}^{N_f}(x+m_j)}{\prod_{l=1}^{N_c}(x-a_l)^2} = \frac{S_k(x)}{(x-a_k)^2} \tag{4.1}$$

Then the prepotential $\mathcal{F}$ is given by

$$\mathcal{F} = \mathcal{F}^{(0)} + \mathcal{F}^{(1)} + \mathcal{F}^{(2)} + O(\bar{\Lambda}^6)$$

with the terms $\mathcal{F}^{(0)}$, $\mathcal{F}^{(1)}$, $\mathcal{F}^{(2)}$ corresponding respectively to the one-loop perturbative contribution, the 1-instanton contribution, and the 2-instanton contribution

$$\begin{aligned}
2\pi i\mathcal{F}^{(0)} =& -\frac{1}{4}\sum (a_k-a_l)^2 \log\frac{(a_k-a_l)^2}{\Lambda^2} + \frac{1}{4}\sum_{j,k}(a_k+m_j)^2\log\frac{(a_k+m_j)^2}{\Lambda^2} \\
2\pi i\mathcal{F}^{(1)} =& \frac{1}{4}\bar{\Lambda}^2\sum_{k=1}^{N_c} S_k(a_k) \\
2\pi i\mathcal{F}^{(2)} =& \frac{1}{16}\bar{\Lambda}^4\big(\sum_{k\neq l}\frac{S_k(a_k)S_l(a_l)}{(a_k-a_l)^2} + \frac{1}{4}\sum_{k=1}^{N_c} S_k(a_k)\partial_{a_k}^2 S_k(a_k)\big)
\end{aligned} \tag{4.2}$$

Here we have ignored quadratic terms in a_k, since they are automatically annihilated by the operator $\mathcal{D}$. We also note that the arguments of [6] only determine $\mathcal{F}$ up to a_k-independent terms, and thus we shall drop all such terms in the subsequent considerations. The formulae (4.2) imply

$$\sum_{k=1}^{N_c} a_k\frac{\partial\mathcal{F}}{\partial a_k} + \sum_{j=1}^{N_f} m_j\frac{\partial\mathcal{F}}{\partial m_j} - 2\mathcal{F} = (N_f-2N_c)\big(\frac{1}{4\pi i}\sum_{k=1}^{N_c} a_k^2 + \mathcal{F}^{(1)} + 2\mathcal{F}^{(2)}\big) \tag{4.3}$$

where all $\bar{\Lambda}^6$ terms have been ignored.

On the other hand, up to a_k-independent terms, the renormalization group equation (3.10) reads

$$\sum_{k=1}^{N_c} a_k\frac{\partial\mathcal{F}}{\partial a_k} + \sum_{j=1}^{N_f} m_j\frac{\partial\mathcal{F}}{\partial m_j} - 2\mathcal{F} = \frac{1}{4\pi i}(N_f-2N_c)\sum_{k=1}^{N_c}\tilde{a}_k^2 \tag{4.4}$$

where we have rewritten $\tilde{s}_2$ as

$$\tilde{s}_2 = -\frac{1}{2}\sum_{k=1}^{N_c}\tilde{a}_k^2 + \frac{\bar{\Lambda}^2}{16}\delta_{N_f,2N_c-1} \tag{4.5}$$

To compare (4.3) with (4.4) we need first to evaluate $\sum_{k=1}^{N_c}\tilde{a}_k^2$ in terms of the renormalized order parameters a_k. Using the formula (3.11) of [6], this can be done routinely

$$a_k = \tilde{a}_k + \frac{\bar{\Lambda}^2}{4}\tilde{\partial}_k\tilde{S}_k(\tilde{a}_k) + \frac{\bar{\Lambda}^4}{64}\tilde{\partial}_k^3\tilde{S}_k(\tilde{a}_k) + O(\bar{\Lambda}^6) \tag{4.6}$$

where we have set $\tilde{\partial}_k = \partial/\partial\tilde{a}_k$, and defined functions $\tilde{S}(x)$, $\tilde{S}_k(x)$ in analogy with (4.1), but with a_k replaced by $\tilde{a}_k$. Inverting $\tilde{a}_k$ in terms of a_k, and rewriting the result in terms of the derivatives $\partial_k = \partial/\partial a_k$ with respect to the renormalized parameters a_k, we find

$$\tilde{a}_k = a_k - \frac{\bar{\Lambda}^2}{4}\partial_k S_k(a_k) - \frac{\bar{\Lambda}^4}{64}\partial_k^3 S_k(a_k)^2 + \frac{\bar{\Lambda}^4}{16}\sum_{l=1}^{N_c}\partial_l S_l(a_l)\partial_k\partial_l S_k(a_l) + O(\bar{\Lambda}^6) \tag{4.7}$$

and hence

$$\begin{aligned}\sum_{k=1}^{N_c}\tilde{a}_2^2 =& \sum_{k=1}^{N_c}a_k^2 - \frac{\bar{\Lambda}^2}{2}\sum_{k=1}^{N_c}a_k\partial_k S_k(a_k) - \frac{\bar{\Lambda}^4}{32}\sum_{k=1}^{N_c}a_k\partial_k^3 S_k(a_k)^2\\ &+ \frac{\bar{\Lambda}^4}{8}\sum_{k,l=1}^{N_c}a_k\partial_l S_l(a_l)\partial_k\partial_l S_k(a_k) + \frac{\bar{\Lambda}^4}{16}\sum_{k=1}^{N_c}(\partial_k S_k(a_k))^2 + O(\bar{\Lambda}^6)\end{aligned} \tag{4.8}$$

Next, we need a number of identities which can be established by contour integrals, in analogy with the identities in Appendix B of [6]

$$\begin{aligned}\sum_{k=1}^{N_c}a_k\partial_k S_k(a_k) =& -\sum_{k=1}^{N_c}S_k(a_k) + \{a_k\text{–independent terms}\}\\ \sum_{k=1}^{N_c}a_k\partial_k^3 S_k(a_k)^2 =& -3\sum_{k=1}^{N_c}\partial_k^2 S_k(a_k)^2\\ \sum_{k,l}a_k\partial_l S_l(a_l)\partial_k\partial_l S_k(a_k) =& -2\sum_{l=1}^{N_c}(\partial_l S_l(a_l))^2 + 2\sum_{k\neq l}\frac{S_k(a_k)S_l(a_l)}{(a_k-a_l)^2}\\ &-\sum_{k\neq l}S_k(a_k)\partial_k^2 S_k(a_k)\end{aligned} \tag{4.9}$$

Using (4.9) we can indeed recast $\sum_{k=1}^{N_c}\tilde{a}_k^2$ as

$$\sum_{k=1}^{N_c}\tilde{a}_k^2 = \sum_{k=1}^{N_c}a_k^2 + \sum_{k=1}^{N_c}\frac{\bar{\Lambda}^2}{2}S_k(a_k) + \frac{\bar{\Lambda}^4}{4}\big(\sum_{k\neq l}\frac{S_k(a_k)S_l(a_l)}{(a_k-a_l)^2} + \frac{1}{4}\sum_{k=1}^{N_c}S_k(a_k)\partial_k^2 S_k(a_k)\big) \tag{4.10}$$

The equality of the two right hand sides in (4.3) and (4.4) follows.

REFERENCES

[1] N. Seiberg and E. Witten, Nucl. Phys. B 426 (1994) 19, hep-th/9407087; Nucl. Phys. B 431 (1994) 484, hep-th/9408099.

[2] A. Klemm, W. Lerche, S. Yankielowicz, and S. Theisen, Phys. Lett. B 344 (1995) 169;
P.C. Argyres and A. Faraggi, Phys. Rev. Lett. 73 (1995) 3931;
P.C. Argyres, R. Plesser, and A. Shapere, Phys. Rev. Lett. 75 (1995) 1699, hep-th/9505100;
J. Minahan and D. Nemeshansky, hep-th/9507032;
U.H. Danielsson and B. Sundborg, Phys. Lett. B 358 (1995) 273, USITP-95-12, UUITP-20/95; hep-th/9504102;
A. Brandhuber and K. Landsteiner, Phys. Lett. B 358 (1995) 73, hep-th/9507008;
M. Alishahiha, F. Ardalan, and F. Mansouri, hep-th/9512005.

[3] A. Hanany and Y. Oz, Nucl. Phys. B 452 (1995) 73, hep-th/9505075
P.C. Argyres and A. Shapere, hep-th/9609175;
A. Hanany, hep-th/9509176.

[4] I.M. Krichever and D.H. Phong, hep-th/9604199, J. of Differential Geometry;

[5] R. Donagi and E. Witten, hep-th/9511101;

[6] E. D'Hoker, I.M. Krichever, and D.H. Phong, hep-th/9609041; hep-th/9609145;

[7] S. Kachru, A. Klemm, W. Lerche, P. Mayr, and C. Vafa, Nucl. Phys. B 459 (1996) 537, hep-th/9508155;

[8] M. Matone, Phys. Lett. B 357 (1995) 342;
G. Bonelli and M. Matone, Phys. Rev. Lett. 76 (1996) 4107;
G. Bonelli and M. Matone, hep-th/9605090;

[9] T. Eguchi and S.K. Yang, Mod. Phys. Lett. A 11 (1996) 131;
J. Sonnenschein, S. Theisen, and S. Yankielowicz, hep-th/9510129.

[10] I.M. Krichever, Comm. Pure Appl. Math. 47 (1994) 437.

[11] A. Gorsky, I.M. Krichever, A. Marshakov, A. Mironov, and A. Morozov, Phys. Lett. B 355 (1995) 466, hep-th/9505035;
H. Itoyama and A. Morozov, hep-th/9512161, hep-th/9601168;
A. Marshakov, hep-th/9602005
E. Martinec and N. Warner, hep-th/9509161, hep-th/9511052;
E. Martinec, hep-th/9510204,
C. Anh and S. Nam, hep-th/9603028,
T. Nakatsu and K. Takasaki, hep-th/9509162;

[12] A. Klemm, W. Lerche, and S. Theisen, hep-th/9505150;
K. Ito and S.K. Yang, hep-th/96

[13] K. Ito and N. Sasakura, SLAC-PUB-KEK-TH-470, hep-th/9602073;

[14] N. Dorey, V. Khoze, and M. Mattis, hep-th/9606199, hep-th/9607202;
Y. Ohta, hep-th/9604051, hep-th/9604059;
A. Klemm, W. Lerche, P. Mayr, C. Vafa, and N. Warner, hep-th/9604034;

[15] D. Finnell and P. Pouliot, Nucl. Phys. **B453** (1995) 225;
A. Yung, hep-th/9605096;
F. Fucito and G. Travaglini, hep-th/9605215.

[16] T. Harano and M. Sato, hep-th/9608060